A THEOREM ON ANALYTIC STRONG MULTIPLICITY ONE

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Abstract

Let K be an algebraic number field, and $\pi = \otimes\pi_v$ an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_K)$ with analytic conductor $C(\pi)$. The theorem on analytic strong multiplicity one established in this note states, essentially, that there exists a positive constant c depending on $\varepsilon > 0, m$, and K only, such that π can be decided completely by its local components π_v with norm $N(v) < c \cdot C(\pi)^{2m+\varepsilon}$.

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1. Introduction

Let K be an algebraic number field, let π and π' be two cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_K)$ with restricted tensor product decompositions $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$. The strong multiplicity one theorem states that if $\pi_v \cong \pi'_v$ for all but finitely many places v, then $\pi = \pi'$. The reader is referred to [1] for history and references in this direction.

The analytic version of the above theorem gives, in terms of the analytic conductor $C(\pi)$ of π defined in (2.14), more precise information on the number of places needed to decide a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A}_K)$. Such an analytic result was first established by Moreno [10], by applying zero-free regions of the Rankin-Selberg L-function of two automorphic representations. To state the result, let $\mathcal{B}_m(Q)$ denote the set of all cuspidal automorphic representations π on $\mathrm{GL}_m(\mathbb{A}_K)$ with analytic conductors $C(\pi)$ less than a large real number Q. Suppose that $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ are in $\mathcal{B}_m(Q)$ with $m \geq 2$. Then, according to [10], there exist positive constants c and d such that, if $\pi_v \cong \pi'_v$ for all finite places v with norm

$$N(v) \leq \begin{cases} cQ^d, & \text{for } m = 2, \\ c\exp(d\log^2 Q), & \text{for } m \geq 3, \end{cases} \tag{1.1}$$

then $\pi = \pi'$.

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Using a different method, Brumley [1] strengthened Moreno's result in (1.1) to

$$N(v) \leq cQ^d, \quad \text{for } m \geq 1, \tag{1.2}$$

where c and d are positive constants depending on m. Brumley's method [1] actually proves that

$$d = \frac{17}{2}m - 4 + \varepsilon \tag{1.3}$$

is acceptable in (1.2). By applying a still different method, the second named author [16] showed essentially that, for all K, the d in (1.2) can be reduced to

$$d = 4m + \varepsilon. \tag{1.4}$$

The purpose of this note is to show that a suitable modification of the argument in [16] actually gives the acceptable value

$$d = 2m + \varepsilon \tag{1.5}$$

in (1.2). To state the result, let $\mathcal{A}_M(Q)$ denote the set of all cuspidal automorphic representations π on $\mathrm{GL}_m(\mathbb{A}_K)$, with $1 \leq m \leq M$, whose analytic conductors $C(\pi)$ are less than a large real number Q. Thus,

$$\mathcal{A}_M(Q) = \bigcup_{m \leq M} \mathcal{B}_m(Q),$$

and the main result of this note can be stated as follows.

Theorem. *Let $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ be in $\mathcal{A}_M(Q)$. Then there exists a constant $c = c(\varepsilon, M, K)$ depending on $\varepsilon > 0, M,$ and K only, such that if $\pi_v \cong \pi'_v$ for all finite places with norm*

$$N(v) < cQ^{2M+\varepsilon}$$

then $\pi = \pi'$.

To prove the Theorem, we will exploit, among other things, Landau's classical idea in [7].

2. Preliminaries on automorphic L-functions

In this section, we summarize some basic properties of automorphic L-functions, which will be used in the proof of the Theorem in §3.

Let K be an algebraic number field of degree $[K : \mathbb{Q}] = l$, with adèle ring $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$, where K_∞ is the product of the Archimedean completions of K, and the ring $\mathbb{A}_{K,f}$ of finite adèles is a restricted direct product of the completions K_v over non-Archimedean places v. Suppose

that π is an automorphic irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_K)$. Then π is a restricted tensor product

$$\pi = \otimes_v \pi_v = \pi_\infty \otimes \pi_f, \tag{2.1}$$

where v runs over all places of K, and π_v is unramified for almost all finite places v. At every finite place v where π_v is unramified we associate a semisimple conjugacy class

$$A_{\pi,v} = \begin{pmatrix} \alpha_{\pi,v}(1) & & \\ & \ddots & \\ & & \alpha_{\pi,v}(m) \end{pmatrix},$$

and define the local L-function for the finite place v as

$$L(s,\pi_v) = \det\left(I - \frac{A_{\pi,v}}{q_v^s}\right)^{-1} = \prod_{j=1}^{m}\left(1 - \frac{\alpha_{\pi,v}(j)}{q_v^s}\right)^{-1} \tag{2.2}$$

where $q_v = N(\mathfrak{p}_v) = N(v)$ is the norm of K_v. It is possible to write the local factors at ramified places v in the form of (2.2) with the convention that some of the $\alpha_{\pi,v}(j)$'s may be zero. The finite part L-function $L(s,\pi_f)$ is defined as

$$L(s,\pi_f) = \prod_{v<\infty} L(s,\pi_v). \tag{2.3}$$

And this Euler product is proved to be absolutely convergent for $\sigma = \Re s > 1$. Also, the Archimedean L-function is defined as

$$L(s,\pi_\infty) = \pi^{-lms/2}\prod_{j=1}^{lm}\Gamma\left(\frac{s+b_\pi(j)}{2}\right). \tag{2.4}$$

The coefficients $\{\alpha_{\pi,v}(j)\}_{1\le j\le m}$ and $\{b_\pi(j)\}_{1\le j\le m}$ are called local parameters of π, respectively, at finite places and at infinite places. For them, a trivial bound states that

$$|\alpha_{\pi,v}(j)| \le \sqrt{p}, \qquad |\Re b_\pi(j)| \le \frac{1}{2}.$$

In connection with (2.1), the complete L-function associated to π is defined by

$$L(s,\pi) = L(s,\pi_\infty)L(s,\pi_f). \tag{2.5}$$

This complete L-function has an analytic continuation and entire, and satisfies the functional equation

$$L(s,\pi) = W_\pi q_\pi^{\frac{1}{2}-s} L(1-s,\tilde{\pi})$$

where $\tilde{\pi}$ is the contragredient of π, W_π a complex number of modulus 1, and q_π a positive integer called the arithmetic conductor of π [3].

Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be automorphic irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_K)$ and $\mathrm{GL}_{m'}(\mathbb{A}_K)$, respectively. The finite part Rankin-Selberg L-function associated to π and π' is defined by

$$L(s, \pi_f \times \tilde{\pi}'_f) = \prod_{v<\infty} L(s, \pi_v \times \tilde{\pi}'_v), \tag{2.6}$$

where

$$L(s, \pi_v \times \tilde{\pi}'_v) = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{\pi,v}(j)\bar{\alpha}_{\pi',v}(j')}{q_v^s}\right)^{-1} \tag{2.7}$$

are finite local L-functions for unramified finite places v, i.e. where π_v and π'_v are both unramified. It can be defined similarly at places v where π_v or π'_v are ramified. This Euler product is proved to be absolutely convergent for $\sigma > 1$, where $L(s, \pi_f \times \tilde{\pi}'_f)$ has a Dirichlet series expression of the form

$$L(s, \pi_f \times \tilde{\pi}'_f) = \sum_{n=1}^{\infty} \frac{a_{\pi \times \tilde{\pi}'}(n)}{n^s}. \tag{2.8}$$

The complete Rankin-Selberg L-function is defined by

$$L(s, \pi \times \tilde{\pi}') = L(s, \pi_\infty \times \tilde{\pi}'_\infty) L(s, \pi_f \times \tilde{\pi}'_f) \tag{2.9}$$

with

$$L(s, \pi_\infty \times \tilde{\pi}'_\infty) = \pi^{-mm'ls/2} \prod_{j=1}^{mm'l} \Gamma\left(\frac{s + b_{\pi\times\tilde{\pi}'}(j)}{2}\right). \tag{2.10}$$

When the infinite place v is unramified for both π and π', we have

$$\{b_{\pi\times\tilde{\pi}'}(j)\}_{1\le j\le mm'} = \{b_\pi(j) + b_{\tilde{\pi}'}(j')\}_{1\le j\le m, 1\le j'\le m'}.$$

By Shahidi [12], [13], [14], [15], the complete L-function $L(s, \pi \times \tilde{\pi}')$ has an analytic continuation to the entire complex plane, and satisfies the functional equation

$$L(s, \pi \times \tilde{\pi}') = W_{\pi\times\tilde{\pi}'} q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s} L(1-s, \tilde{\pi} \times \pi') \tag{2.11}$$

where $W_{\pi\times\tilde{\pi}'}$ is a complex constant of modulus 1, and $q_{\pi\times\tilde{\pi}'}$ is a positive integer. By Jacquet-Shalika [6] and Moeglin-Waldspurger [9], we know that $L(s, \pi \times \tilde{\pi}')$ is holomorphic when $\pi' \neq \pi \otimes |\det|^{i\tau}$ for any $\tau \in \mathbb{R}$, and the only poles of $L(s, \pi \times \tilde{\pi}')$ are simple poles at $s = i\tau_0$ and

$1 + i\tau_0$, when $m = m'$ and $\pi' = \pi \otimes |\det|^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}$. Finally, by Gelbart-Shahidi [4], $L(s, \pi \times \tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in the vertical strips.

Following Iwaniec-Sarnak [5], we define the analytic conductors of $L(s, \pi)$ and $L(s, \pi \times \tilde{\pi}')$, respectively, as

$$C(\pi; t) = q_\pi \prod_{j=1}^{ml} (1 + |it + b_\pi(j)|), \tag{2.12}$$

and

$$C(\pi, \tilde{\pi}'; t) = q_{\pi \times \tilde{\pi}'} \prod_{j=1}^{mm'l} (1 + |it + b_{\pi \times \tilde{\pi}'}(j)|). \tag{2.13}$$

Setting $t = 0$ in the above definitions, we write

$$C(\pi) = C(\pi; 0), \qquad C(\pi, \tilde{\pi}') = C(\pi, \tilde{\pi}'; 0), \tag{2.14}$$

which are called, respectively, the analytic conductors of π and of $\pi \times \tilde{\pi}'$. They satisfy the inequality

$$C(\pi, \tilde{\pi}') \le C(\pi)^{m'} C(\pi')^m, \tag{2.15}$$

which follows from Bushnell-Henniart [2] or Ramakrishnan-Wang [11].

3. Proof of the Theorem

Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be in $\mathcal{A}_M(Q)$. If they are twisted equivalent, i.e. $\pi = \pi' \otimes |\det|^{i\tau}$ for some $\tau \in \mathbb{R}^\times$, then $\pi_v \not\cong \pi'_v$ for all finite places v with at most one exception. We may therefore suppose that, for all $\tau \in \mathbb{R}$,

$$\pi \ne \pi' \otimes |\det|^{i\tau}. \tag{3.1}$$

To compare π with π', we form the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi}')$, and exploit its Dirichlet series expansion (2.8), which holds for $\sigma > 1$. By (3.1), the functions $L(s, \pi \times \tilde{\pi}')$ and $L(s, \pi_f \times \tilde{\pi}'_f)$ are entire fucntions in the whole complex plane. Define

$$S(x; \pi, \tilde{\pi}') = \sum_{n=1}^{\infty} a_{\pi \times \tilde{\pi}'}(n) w\left(\frac{n}{x}\right), \tag{3.2}$$

where $w(x)$ is a non-negative real valued function of C_c^∞ with compact support in $[0,3]$, and we may specify

$$w(x) = \begin{cases} 0, & \text{for } x \notin [0,3], \\ e^{-\frac{1}{x}}, & \text{for } x \in (0,1], \\ e^{-\frac{1}{3-x}}, & \text{for } x \in [2,3) \end{cases}$$

as in [16, §3]. Thus, for any positive integer k, the derivative $w^{(k)}(x)$ has exponential decay as $x \to 0$ or 3. Consequently, the Mellin transform

$$W(s) = \int_0^\infty w(x) x^{s-1} dx$$

is an analytic function of s; if $\sigma < -1$ then

$$W(s) \ll_{A,\sigma} \frac{1}{(1+|t|)^A}$$

for arbitrary $A > 0$, by repeated partial integration. By Mellin inversion,

$$w(x) = \frac{1}{2\pi i} \int_{(2)} W(s) x^{-s} ds,$$

where (c) means the vertical line $\sigma = c$. Inserting this back to (3.2), and then using Dirichlet series expansion (2.8), we have

$$\begin{aligned} S(x; \pi, \tilde{\pi}') &= \frac{1}{2\pi i} \sum_{n=1}^\infty a_{\pi\times\tilde{\pi}'}(n) \int_{(2)} W(s) \left(\frac{n}{x}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(2)} x^s W(s) L(s, \pi_f \times \tilde{\pi}'_f) ds, \end{aligned}$$

where the interchange of summation and integral is guaranteed by the absolute convergence of (2.8) on the line $\sigma = 2$. A pre-convexity bound like

$$L(s, \pi_f \times \tilde{\pi}'_f) \ll C(\pi, \tilde{\pi}'; t)^B,$$

where $B > 0$ is some constant, can be obtained by standard method, as pointed out in [1, §1]. Since $W(s)$ is rapid decay and $L(s, \pi \times \tilde{\pi}')$ is entire, we may shift the contour above to the vertical line $\sigma = -H$, getting

$$S(x; \pi, \tilde{\pi}') = \frac{1}{2\pi i} \int_{(-H)} x^s W(s) L(s, \pi_f \times \tilde{\pi}'_f) ds, \tag{3.3}$$

where $H > 1$ is a large real number to be specified later.

We are going to apply the functional equation (2.11) to (3.3). To this end, we rewrite (2.11) as

$$L(s,\pi_f\times\tilde{\pi}'_f)=W_{\pi\times\tilde{\pi}'}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s}G(s)L(1-s,\tilde{\pi}_f\times\pi'_f), \tag{3.4}$$

where

$$G(s)=\frac{L(1-s,\tilde{\pi}_\infty\times\pi'_\infty)}{L(s,\pi_\infty\times\tilde{\pi}'_\infty)}. \tag{3.5}$$

We need to estimate $G(s)$ on the line $\sigma=-H$, avoiding the poles of the nominator of $G(s)$. This will be done in the Lemma in §4, which asserts that, for every large positive integer n, there is an $H\in[n,n+1]$ such that, on the vertical line $\sigma=-H$,

$$G(-H+it)\ll_{H,M,K}(1+|t|)^{mm'l(\frac{1}{2}+H)}\prod_{j=1}^{mm'l}(1+|b_{\pi\times\tilde{\pi}'}(j)|)^{\frac{1}{2}+H}. \tag{3.6}$$

Now we apply the functional equation (3.4),

$$\begin{aligned}
S(x;\pi,\pi') &= \frac{1}{2\pi i}\int_{(-H)}x^sW(s)W_{\pi\times\tilde{\pi}'}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s}G(s)L(1-s,\tilde{\pi}_f\times\pi'_f)ds\\
&= \frac{1}{2\pi i}\int_{(-H)}x^sW(s)W_{\pi\times\tilde{\pi}'}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s}G(s)\left(\sum_{n=1}^{\infty}\frac{a_{\tilde{\pi}\times\pi'}(n)}{n^{1-s}}\right)ds\\
&= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{a_{\tilde{\pi}\times\pi'}(n)}{n^{1+H}}\int_{(-H)}x^sW(s)W_{\pi\times\tilde{\pi}'}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s}G(s)n^{s+H}ds.
\end{aligned}$$

Here the interchange of summation and integral is guaranteed by the absolute convergence of the Dirichlet series as well as the rapid decay of $W(s)$. Using these facts again, and inserting (3.6) into the last integral, we get

$$\begin{aligned}
S(x;\pi,\tilde{\pi}') &\ll_{H,M,K} \int_{(-H)}\left|x^sW(s)W_{\pi\times\tilde{\pi}'}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}-s}G(s)ds\right|\\
&\ll_{H,M,K} x^{-H}q_{\pi\times\tilde{\pi}'}^{\frac{1}{2}+H}\prod_{j=1}^{mm'l}(1+|b_{\pi\times\tilde{\pi}'}(j)|)^{\frac{1}{2}+H}\\
&= x^{-H}C(\pi,\tilde{\pi}')^{\frac{1}{2}+H}.
\end{aligned} \tag{3.7}$$

This upper bound corresponds to [16, (10)].

To establish the Theorem, we need a lower bound for $S(x;\pi,\tilde{\pi}')$. Under the assumption of (3.1), we further suppose that $\pi_v \cong \pi'_v$ for all finite places with norm $N(v) < x$. Then

$$S(x;\pi,\tilde{\pi}') = S(x;\pi,\tilde{\pi}). \tag{3.8}$$

A lower bound for $S(x;\pi,\tilde{\pi})$ is obtained in [1, Lemma 2]; see also [16, Propsition 4]. Thus, we have

$$S(x;\pi,\tilde{\pi}) \gg \frac{x^{\frac{1}{m}}}{\log x}. \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we get

$$\frac{x^{\frac{1}{m}}}{\log x} \ll S(x;\pi,\tilde{\pi}') \ll_{H,M,K} x^{-H} C(\pi,\tilde{\pi}')^{\frac{1}{2}+H}.$$

One therefore has

$$x < c \cdot C(\pi,\tilde{\pi}')^{\frac{H+1/2}{H+1/m}},$$

where c is a constant depending on H, M, K. Taking H sufficiently large, this becomes

$$x < c \cdot C(\pi,\tilde{\pi}')^{1+\varepsilon},$$

and now the constant c depends on ε, M, K. The assertion of the theorem finally follows from this and (2.15).

4. An Estimate for $G(s)$

In this section, we give an estimate for $G(s)$ defined as in (3.5) on a vertical line $s = -H + it$, where H is a large real number to be decided suitably. Recall Stirling's formula that

$$|\Gamma(\sigma + it)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} \left(1 + O_{\sigma,\delta}\left(\frac{1}{1+|t|}\right)\right),$$

which holds for $s = \sigma + it$ away from all poles of $\Gamma(s)$ by at least $\delta > 0$; note that the implied constant depends on σ and δ.

To do this, we should first locate the poles of the nominator of $G(s)$, i.e. poles of

$$L(1-s, \tilde{\pi}_\infty \times \pi'_\infty) = \pi^{-mm'ls/2} \prod_{j=1}^{mm'l} \Gamma\left(\frac{1 - s + b_{\tilde{\pi}\times\pi'}(j)}{2}\right), \tag{4.1}$$

according to (2.10). These poles are easily to be seen as

$$P_{n,j} = 2n + 1 + b_{\tilde{\pi}\times\pi'}(j), \quad n = 0, 1, 2, \cdots, \quad j = 1, \cdots, mm'l.$$

As in [8], we let $\mathbb{C}(m,m')$ be the complex plane with the discs

$$|s-P_{n,j}|<\frac{1}{8mm'l},\quad n=0,1,2,\cdots,\quad j=1,\cdots,mm'l$$

excluded. Thus, for any $s\in\mathbb{C}(m,m')$, the quantity

$$\frac{1-s+b_{\tilde{\pi}\times\pi'}(j)}{2}$$

is away from all poles of $\Gamma(s)$ by at least $1/(16mm'l)$, and therefore Stirling's formula applies to (4.1). Of course, Stirling's formula also applies to the denominator of $G(s)$. Writing $b_{\pi\times\tilde{\pi}'}(j)=u(j)+iv(j)$ and $s=\sigma+it\in\mathbb{C}(m,m')$, we have

$$\begin{aligned} G(s) &= \pi^{-\frac{mm'l}{2}+mm'ls}\prod_{j=1}^{mm'l}\frac{\Gamma\left(\frac{1-s+b_{\tilde{\pi}\times\pi'}(j)}{2}\right)}{\Gamma\left(\frac{s+b_{\pi\times\tilde{\pi}'}(j)}{2}\right)} \\ &\ll_{\sigma,M,K} \prod_{j=1}^{mm'l}\frac{|t+v(j)|^{\frac{1-\sigma+u(j)}{2}-\frac{1}{2}}}{|t+v(j)|^{\frac{\sigma+u(j)}{2}-\frac{1}{2}}} \\ &\ll_{\sigma,M,K} \prod_{j=1}^{mm'l}|t+v(j)|^{\frac{1}{2}-\sigma}, \end{aligned}$$

where we have used $\{b_{\tilde{\pi}\times\pi'}(j)\}_{1\le j\le mm'l}=\{\overline{b_{\pi\times\tilde{\pi}'}(j)}\}_{1\le j\le mm'l}$. It follows that, for $\sigma<1/2$,

$$G(s)\ll_{\sigma,M,K}(1+|t|)^{mm'l(\frac{1}{2}-\sigma)}\prod_{j=1}^{mm'l}(1+|v(j)|)^{\frac{1}{2}-\sigma},\tag{4.2}$$

which can be written as

$$G(s)\ll_{\sigma,M,K}(1+|t|)^{mm'l(\frac{1}{2}-\sigma)}\prod_{j=1}^{mm'l}(1+|b_{\pi\times\tilde{\pi}'}(j)|)^{\frac{1}{2}-\sigma}.\tag{4.3}$$

Now we give a remark about the structure of $\mathbb{C}(m,m')$. For $j=1,\cdots,mm'l$, denote by $\beta(j)$ the fractional part of $v(j)$. In addition we let $\beta(0)=0$ and $\beta(mm'l+1)=1$. Then all $\beta(j)\in[0,1]$, and hence there exist $\beta(j_1),\beta(j_2)$ such that $\beta(j_2)-\beta(j_1)\ge 1/(3mm'l)$ and there is no $\beta(j)$ lying between $\beta(j_1)$ and $\beta(j_2)$. It follows that the strip $S_0=\{s:\beta(j_1)+1/(8mm'l)\le\Re s\le\beta(j_2)-1/(8mm'l)\}$ is contained in $\mathbb{C}(m,m')$. Consequently, for all $n=0,1,2,\cdots$, the strips

$$S_n=\left\{s:-n+\beta(j_1)+\frac{1}{8mm'l}\le\Re s\le -n+\beta(j_2)-\frac{1}{8mm'l}\right\}$$

are subsets of $\mathbb{C}(m, m')$. Therefore, for each $n \geq 1$, one can choose a vertical line $\sigma = -H$ lying in the strip S_n, and therefore (4.3) holds on the line $\sigma = -H$. This proves the following result.

Lemma. *Let $G(s)$ be as in (3.5). Then for each $n \geq 1$, there is an $H \in [n, n+1]$, such that on the line $\sigma = -H$ we have*

$$G(-H+it) \ll_{H,M,K} (1+|t|)^{mm'l(\frac{1}{2}+H)} \prod_{j=1}^{mm'l} (1+|b_{\pi\times\tilde{\pi}'}(j)|)^{\frac{1}{2}+H}.$$

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